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Pairwise balance and invariant measures for generalized exclusion processes

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Abstract. We characterize the steady state of a driven diffusive lattice gas in which each site holds several particles, and the dynamics is activated and asymmetric. Using a quantum Hamiltonian formalism, we show that for arbitrary transition rates the model has product invariant measure. In the steady state, a pairwise balance condition is shown to hold. Configurations $\underline{n}''$ and $\underline{n}'$ leading respectively into and out of a given configuration $\underline{n}$ are matched in pairs so that the flux of transitions from $\underline{n}''$ to $\underline{n}$ is equal to the flux from $\underline{n}$ to $\underline{n}'$. Pairwise balance is more general than the condition of detailed balance and holds in the non-equilibrium steady state of a number of stochastic models.

1. Introduction

We consider a driven diffusive lattice gas in one dimension that generalizes the simple exclusion process introduced by Spitzer [1, 2]. This model is described as follows. The system consists of indistinguishable particles on a lattice which have the properties:

(a) there can be at most $N \geqslant 1$ particles at each lattice site; and

(b) each particle interacts only with other particles at the same site.

During the time interval $(t, t + \mathrm{d}t)$, a particle which is at site i makes a transition to the closest site $i + n$ which is not fully occupied at that time. This process occurs with probability $c_k \mathrm{d}t$, where $k \leqslant N$ is the number of particles at i at time t and $0 < c_k < \infty$.

We term this process drop-push dynamics [3], which may be visualized as particles moving in one direction (say rightward) on a one-dimensional lattice, with each lattice site being a trap that can hold a *maximum* of N particles. Escape from a trap is activated and the rate of escape depends on the number of particles already in there. We also imagine that the particle that hops out of a site i is the topmost particle at that site, whereas when a particle drops into a trap it goes to the bottom. Should a trap be full, then a particle that hops into that site pushes the topmost particle out into the next site, thus effectively bouncing with unit probability to the next site on the right.

One motivation to study models such as the above arises from physical considerations such as the nature of transport in tenuously connected media. A physical realization of the drop-push model occurs, for example, in the field-induced transport of particles in a random

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medium such as the percolation cluster [3]: the field determines the asymmetry in transition rates, a path on the backbone provides the effective 1D lattice on which the particles move and backbends [4] are the traps from which activated escape takes place. This and other interacting particle systems have recently begun to attract much attention, as they provide examples of systems far from equilibrium which are amenable to some degree of analysis [5]. For the case $N = 1$, the above drop-push model is related to simple asymmetric exclusion models [2] which have been extensively studied [6–8], and for which several results are known. The case $N = 1$ also corresponds to the dynamics of a Toom model interface in the low noise limit [9], where it is known that the steady state is characterized by product measure.

In the present paper we show that the steady state of the lattice gas with drop-push dynamics can also be completely determined. We define the system in terms of a master operator technique—the 'quantum Hamiltonian' formalism—to show that for arbitrary $N \geqslant 1$ and arbitrary c_k, the invariant distribution has product measure. This is detailed in section 2. In section 3 we define the condition of pairwise balance; namely that in the steady state, with every transition out of a particular configuration is uniquely associated an equal-flux transition leading into that configuration. This generalization of the well known condition of detailed balance holds for drop-push dynamics and several other stochastic lattice gas models. This is followed by a brief summary in section 4.

2. Invariant measures

Consider the finite lattice $S = \{0, 1, 2, \ldots, L-1\}$ with periodic boundary conditions, i.e. the one-dimensional integer lattice with integers taken modulo L. There can be up to N particles at each site and hence the state space of the process is $X = \{0, 1, \ldots, N\}^S$. A given configuration $\underline{n}$ is a specification of all site occupancies k_i where $0 \leqslant i \leqslant L-1$ and $0 \leqslant k \leqslant N$. We shall denote the probability of finding the configuration $\underline{n}$ at time t by $f(\underline{n}; t)$ with $\sum_{\underline{n}\in X} f(\underline{n}; t) = 1$ and define the process in terms of a master equation satisfied by $f(\underline{n}; t)$. As described in the introduction, transitions to other configurations, such as

$$(\ldots, m_i = k, m_{i+1} = l, \ldots) \rightarrow (\ldots, m_i = k-1, m_{i+1} = l+1, \ldots) \tag{1}$$

or, if sites $i+1$ up to $i+r-1$ are *all* fully occupied,

$$(\ldots, m_i = k, N, N, \ldots, m_{i+r} = l, \ldots) \rightarrow (\ldots, m_i = k-1, N, N, \ldots, m_{i+r} = l+1, \ldots) \tag{2}$$

both occur at rate c_k, where we define $c_0 = 0$. In order to write a master equation for the process we follow a route that has been successfully applied (particularly in the past few years) in the study of one-dimensional stochastic dynamics, the quantum Hamiltonian formulation.

We assign to each element $\underline{n} \in X$ a *basis vector* $|\underline{n}\rangle$ in a $(N+1)^L$-dimensional vector space which we shall also denote by X. In addition to $|\underline{n}\rangle$ we shall define the transposed vectors $\langle\underline{n}|$ which endow X with an orthonormal basis and a scalar product written $\langle\cdot|\cdot\rangle$ and defined by $\langle\underline{n}|\underline{n}'\rangle = \delta_{\underline{n},\underline{n}'}$. The simple form of the state space suggests the tensor basis,

$|\underline{n}\rangle \equiv |k_1) \otimes |k_2) \otimes \cdots \otimes |k_L)$ where the single site vectors

$$|k) \equiv \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tag{3}$$

denoting the occupation by k particles have a 1 in the $(k+1)$th position in the column and zero elsewhere. In this way the probability distribution $f(\underline{n};t)$ is mapped onto a state vector

$$|f(t)\rangle \equiv \sum_{\underline{n}\in X} f(\underline{n};t)|\underline{n}\rangle \tag{4}$$

i.e. $f(\underline{n};t) = \langle\underline{n}|f(t)\rangle$. By defining

$$\langle s| \equiv \sum_{\underline{n}\in X} \langle\underline{n}| \tag{5}$$

conservation (and normalization) of probability reads $\langle s|f(t)\rangle = \sum_{\underline{n}\in X} f(\underline{n};t) = 1$.

In what follows we need 'local operators' Q_i acting non-trivially only on site i of the lattice, i.e. matrices acting on $X = (C^{N+1})^{\otimes L}$ which have the form $Q_i \equiv 1 \otimes 1 \otimes \cdots \otimes Q \otimes \cdots \otimes 1$, where Q is a $(N+1)\times(N+1)$ matrix at the ith position in the tensor product and 1 is the $(N+1)$-dimensional unit matrix. In particular we shall use $E_i^{\alpha,\beta}$ where $E^{\alpha,\beta}$ is the $(N+1)\times(N+1)$ matrix with matrix elements $(E^{\alpha,\beta})_{p,q} = \delta_{\alpha,p}\delta_{\beta,q}$ $(1 \leqslant \alpha,\beta,p,q \leqslant N+1)$. These matrices have a simple interpretation in terms of the stochastic process: $E_i^{\alpha,\alpha} \equiv P_i^{(\alpha-1)}$ projects on states which have $\alpha-1$ particles on site i, whereas $E_i^{\alpha,\beta}$ for $\alpha \neq \beta$ turns a state with $\beta-1$ particles on site i into a state with $\alpha-1$ particles. Particle occupancies on other sites remain unchanged.

Now we are in a position to define the process in terms of a linear operator H acting on $X = (C^{N+1})^{\otimes L}$:

$$\frac{\partial}{\partial t}|f(t)\rangle = -H|f(t)\rangle \tag{6}$$

where the 'quantum Hamiltonian operator' H for the process described above is given by

$$H = -\sum_{i=1}^{L}\sum_{r=1}^{L-1} u_i(r). \tag{7}$$

The hopping of a particle with rate $0 < c_k < \infty$ from site i with k particles to site $i+r$ with $k' < N$ particles with fully occupied intermediate sites $(i+1,\ldots,i+r-1)$ is encoded in the matrices

$$u_i(r) \equiv A_i \prod_{j=i+1}^{i+r-1} P_j^{(N)} B_{i+r} - S_i \prod_{j=i+1}^{i+r-1} P_j^{(N)} T_{i+r}. \tag{8}$$

In this expression $P_j^{(N)}$ are the projectors on fully occupied lattice sites defined in the preceding paragraph and

$$A = \sum_{k=2}^{N+1} c_{k-1} E^{k-1,k} \tag{9}$$

$$B = \sum_{k=1}^{N} E^{k+1,k} \tag{10}$$

$$S = \sum_{k=1}^{N} c_k P^{(k)} \tag{11}$$

$$T = \sum_{k=0}^{N-1} P^{(k)}. \tag{12}$$

This is the complete definition of the process. The master equation satisfied by $f(\underline{n};t)$ may be obtained in its usual form from (4) and (6) by taking the scalar product with $\langle \underline{n}|$. The resulting expression is lengthy and unilluminating and we shall not write it down here. Note that the solution to the master equation (6) in terms of an initial distribution $|f\rangle$ at time $t = 0$ is given by $|f(t)\rangle = \exp(-Ht)|f\rangle$ $(t \geqslant 0)$. If (and only if) $|f\rangle$ is a (right) eigenvector of H with eigenvalue zero then the measure defined by this state is an invariant measure. H is constructed such that the real part of all eigenvalues of H is larger than or equal to zero. Its matrix elements $(H)_{p,q}$ satisfy $(H)_{p,p} \geqslant 0\ \forall p$, $(H)_{p,q} \leqslant 0\ \forall p, q$ with $p \neq q$ and $\sum_p H_{p,q} = 0\ \forall q$, i.e. the sum of all entries in each column adds up to zero.

Now we show the following.

The product measure defined by the state vector $|\mu\rangle = |\mu)^{\otimes L}$ where $\mu \geqslant 0$,

$$|\mu) = \frac{1}{Z}\begin{pmatrix} \lambda_0 \\ \lambda_1\mu \\ \lambda_2\mu^2 \\ \vdots \\ \lambda_N\mu^N \end{pmatrix} \qquad \lambda_0 = 1,\ \lambda_k = \frac{c_1^k}{c_1c_2\cdots c_k} \tag{13}$$

and $Z = \sum_{k=0}^{N} \lambda_k\mu^k \geqslant 1$ is an invariant measure of the process, i.e. the vector $|\mu\rangle$ is a right eigenstate of H with eigenvalue 0 and $\langle s|\mu\rangle = 1$.

We first prove $\langle s|\mu\rangle = 1$. The vector $\langle s|$ defined in (5) may be written in tensor form as $\langle s| = (\sum_{m=0}^{N}(m|)^{\otimes L}$, while $|\mu\rangle = (\sum_{k=0}^{N}\lambda_k\mu^k|k)/Z)^{\otimes L}$. Since $(m|k) = \delta_{m,k}$ one finds $\langle s|\mu\rangle = (\sum_{k=0}^{N}\lambda_k\mu^k)^L/Z^L = 1$.

The proof that $H|\mu\rangle = 0$ is less trivial except for $\mu = 0$ and $\mu = \infty$ respectively, in which cases the lattice is completely empty (respectively fully occupied) and therefore trivially stationary. Thus, in what follows we shall assume $\mu \neq 0, \infty$. First, we note the following identities:

$$A|\mu) = \mu\sum_{k=0}^{N-1}\frac{\lambda_{k+1}}{\lambda_k}c_{k+1}P^{(k)}|\mu) \tag{14}$$

$$B|\mu) = \mu\sum_{k=1}^{N}\frac{\lambda_{k-1}}{\lambda_k}P^{(k)}|\mu) \tag{15}$$

which are simple consequences of the definition of the quantities involved. Therefore, one has

$$\begin{aligned} u_i(r)|\mu\rangle &= \Bigg\{\sum_{k=1}^{N}\sum_{l=0}^{N-1}\frac{\lambda_{k-1}}{\lambda_k}\frac{\lambda_{l+1}}{\lambda_l}c_{l+1}P_i^{(l)}\Bigg(\prod_{j=i+1}^{i+r-1}P_j^{(N)}\Bigg)P_{i+r}^{(k)} \\ &\quad -\sum_{k=1}^{N}\sum_{l=0}^{N-1}c_kP_i^{(k)}\Bigg(\prod_{j=i+1}^{i+r-1}P_j^{(N)}\Bigg)P_{i+r}^{(l)}\Bigg\}|\mu\rangle \qquad (16) \\ &\equiv \hat{u}_i(r)|\mu\rangle \qquad (17) \end{aligned}$$

where $\hat{u}_i(r)$ is defined by the expression inside the curly brackets. Shifting summation indices and using $\sum_{k=0}^{N} P_i(k) = 1$ yields

$$\hat{u}_i(r) = \sum_{k=1}^{N} c_k \Bigg[\Bigg(\prod_{j=i+1}^{i+r-1} P_j^{(N)} \Bigg) P_{i+r}^{(k)} - \Bigg(\prod_{j=i}^{i+r-1} P_j^{(N)} \Bigg) P_{i+r}^{(k)} + P_i^{(k)} \Bigg(\prod_{j=i+1}^{i+r} P_j^{(N)} \Bigg) - P_i^{(k)} \Bigg(\prod_{j=i+1}^{i+r-1} P_j^{(N)} \Bigg) \Bigg] \tag{18}$$

and therefore

$$\begin{aligned} H|\mu\rangle &= \sum_{i=1}^{L} \sum_{r=1}^{L-1} \hat{u}_i(r)|\mu\rangle \\ &= \sum_{k=1}^{N} c_k \sum_{i=1}^{L} \Bigg\{ \sum_{r=2}^{L-1} \prod_{j=i}^{i+r-2} P_j^{(N)} P_{i+r-1}^{(k)} - \sum_{r=1}^{L-1} \prod_{j=i}^{i+r-1} P_j^{(N)} P_{i+r}^{(k)} \\ &\quad + \sum_{r=1}^{L-1} P_i^{(k)} \prod_{j=i+1}^{i+r} P_j^{(N)} - \sum_{r=2}^{L-1} P_i^{(k)} \prod_{j=i+1}^{i+r-1} P_j^{(N)} \Bigg\} |\mu\rangle \\ &= \sum_{k=1}^{N} c_k \sum_{i=1}^{L} \Bigg[P_i^{(k)} \prod_{j=i+1}^{i+L-1} P_j^{(N)} - \prod_{j=i}^{i+L-2} P_j^{(N)} P_{i+L-1}^{(k)} \Bigg] |\mu\rangle \\ &= 0. \end{aligned} \tag{19}$$

In this part of the calculation only summation indices were shifted leading to a cancellation of all terms $\hat{u}_i(r)$ in the double sum over i and r.

The occurrence of the parameter μ reflects conservation of the total number of particles in the process. Expanding $Z^L|\mu\rangle$ in a power series in μ as $Z^L|\mu\rangle = \sum_{M=0}^{NL} \mu^M |\tilde{M}\rangle$ and normalizing $|M\rangle \equiv |\tilde{M}\rangle / \langle s|\tilde{M}\rangle$ yields invariant measures $|M\rangle$ characterizing the process with a fixed number of particles M.

The same calculation could be repeated for the reflected process where particles hop to the nearest available site to their *left* with the same rates c_k. This would lead to the same invariant measure which is therefore invariant also for the combined process where particles are allowed to hop to the right or to the left with rates $a_L c_k$ and $a_R c_k$ respectively and with an arbitrary, but k-independent ratio a_L/a_R.

The product invariant measure in equation (13) also appears to obtain in a variety of stochastic lattice gas models with dynamical rules quite different from the drop-push case presently under consideration—for example, Spitzer's zero-range process [1] wherein particles can hop from a given site to any other site. Another case is the stationary state of an independent site model with a master equation of the form $\dot{P}_n = c_{n+1} P_{n+1} - c_n P_n + \mu c_1 (P_{n-1} - P_n)$. This system will also have the same product measure, although the underlying dynamics is, of course, quite distinct.

3. Pairwise balance

In this section we define a pairwise balance condition which is a generalization of the condition of detailed balance for equilibrium systems and show that a lattice gas obeying drop-push dynamics satisfies this condition.

The condition for stationarity is that the total flux into each configuration $\underline{n}$ is equal to the flux out of the configuration,

$$\sum_{\underline{n}' \in X} w(\underline{n} \to \underline{n}') f_\mu(\underline{n}) = \sum_{\underline{n}'' \in X} w(\underline{n}'' \to \underline{n}) f_\mu(\underline{n}'') \tag{20}$$

where $w(\underline{n} \to \underline{n}')$ is a transition rate and $f_\mu(\underline{n})$ is the stationary value of $f(\underline{n}; t)$. A sufficient condition for this is as follows.

With every configuration $\underline{n}'$ that can be reached by an elementary transition from configuration $\underline{n}$ is associated a unique configuration $\underline{n}''$ with the property that in the steady state

$$w(\underline{n}'' \to \underline{n}) f_\mu(\underline{n}'') = w(\underline{n} \to \underline{n}') f_\mu(\underline{n}). \tag{21}$$

This association is 1–1 and exhaustive: the identification of a particular influx with a particular outflux is the condition required for *pairwise balance* to hold.

The familiar condition of detailed balance [10] is obtained as the special case in which the configurations $\underline{n}'$ and $\underline{n}''$ are the same. Detailed balance holds in equilibrium systems, while pairwise balance holds also for the non-equilibrium steady states in a class of stochastic models of the type under consideration here. (While we have verified its applicability to several models, the complete range of validity of pairwise balance remains to be explored.)

Now we show that the condition of pairwise balance is satisfied for drop-push dynamics. Suppose the transition $\underline{n} \to \underline{n}'$ involves transferring a particle from site i (with occupation $m_i = k$) to site $i + r$ (with $m_{i+r} = \ell$), as in equation (2), where each of the $(r - 1)$ sites in between is full ($m_p = N$ for $p = i, i + 1, \dots, i + r - 1$). Suppose that the $(q - 1)$ sites preceding i are also full ($m_p = N$ for $p = i - 1, \dots, i - q + 1$), and that $m_{i-q} = j < N$. Configuration $\underline{n}''$ is defined as that which has the same occupations as configuration $\underline{n}$, for all sites except the pair of sites $i - q$ and i; at these sites, $m_{i-q} = j + 1$ and $m_i = k - 1$.

The transition from $\underline{n}''$ to $\underline{n}$, i.e.

$$(\dots, m_{i-q} = j + 1, N, N, \dots, m_i = k - 1, \dots) \to (\dots, m_{i-q} = j, N, N, \dots, m_i = k, \dots) \tag{22}$$

occurs at rate $w(\underline{n}'' \to \underline{n}) = c_{j+1}$ while $w(\underline{n} \to \underline{n}') = c_k$. Writing $f_\mu(\underline{n})$ as $\prod_p \lambda_p$, the pairwise balance condition equation (21) reduces to

$$c_{j+1}\lambda_{j+1}\lambda_{k-1} = c_k\lambda_j\lambda_k. \tag{23}$$

This condition is satisfied, in view of equation (13).

While we have considered the transition caused by a rightward hop, the argument holds, by reflection, for leftward hops also. Thus, drop-push dynamics satisfies the condition of pairwise balance.

4. Summary

Here we have examined a recently introduced stochastic lattice gas model which is of relevance in the study of a variety of transport phenomena. In this system, the Markov chain is irreducible: each configuration can be reached from any other configuration and the dynamics is ergodic. We have determined the unique invariant measure of the steady state through a quantum Hamiltonian formalism. In this steady state the probability $p_\mu(k)$

of finding $0 \leqslant k \leqslant N$ particles on a site in the lattice is $p_\mu(k) = \lambda_k \mu^k$, where $0 \leqslant \mu \leqslant \infty$ is a free parameter determining the average density of the lattice gas (cf equation (13)). Expanding the unnormalized measure in powers of μ leads to steady states with a fixed number of particles. The probability that a hopping process occurs depends only on the number of particles on the site from which a particle hops away, but not on the overall configuration of the system. Unlike the simple exclusion process, where hops are attempted to a nearest neighbour site and are sucessful only if that site is vacant, in the present model *every* attempted hop is successful. As a result it is the *on-site* interaction which determines the steady state and not the random motion leading to it [1]. In this respect this model bears some similarity to the problem of flows in networks of waiting lines [11].

We have introduced the notion of pairwise balance, which is a generalization of detailed balance to non-equilibrium situations, and shown that the condition is satisfied in the steady state of the model. It is interesting to note that a variety of non-equilibrium systems with stochastic dynamics do in fact also obey this condition—examples are the asymmetric exclusion process [7], its extensions to longer range hopping [12], and Abelian sandpile models [13].

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